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Using Number Lines to Solve Math Word Problems: A Strategy for Students with Learning Disabilities

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Students with learning disabilities (LD) consistently struggle with word problem solving in mathematics classes. This difficulty has made curricular, state, and national tests particularly stressful, as word problem solving has become a predominant feature of such student performance assessments. Research suggests that students with LD perform poorly on word problem-solving items due primarily to deficits in problem representation. Therefore, it is imperative that teachers provide these students with supplemental problem-solving instruction that specifically targets the development of representational strategies. This article describes how one representational strategy, using number lines, can be used to model word problems as part of a comprehensive problem-solving intervention to improve the conceptual understanding of math word problems and, subsequently, the problem-solving performance of students with LD.

Mrs. Wilson, a middle school math teacher, presented her inclusive sixth grade class with the following math story problem from their textbook (Boswell, Kanold, Larson, & Stiff, 2004, p. 43): "The mall is 12 miles from your home. Your school is one third of the way from your home to the mall. The library is one fourth of the way from the school to the mall. How far is the library from home?" Mrs. Wilson followed the textbook's guidelines and instructed her students to draw a picture and then use the picture to solve the problem. After the students completed this task, Mrs. Wilson was surprised to note that few students had arrived at the correct answer, and most students, particularly those with LD, had drawn a detailed picture of the buildings. None of her students showed the problem information and how they related, such as that the school is closer to home than to the mall, or that the library is located between the school and the mall. Upon questioning by Mrs. Wilson, very few students were able to justify their solution approach by referring to their representations.

Solving mathematical word problems is a complex procedure, involving several cognitive processes beyond computing a mathematical equation, which in itself requires specific procedural and conceptual knowledge. To solve a word problem, students must actively understand the problem information, generate and maintain a mental model of the problem based on its components, and finally, decide on a viable solution path based on that model (Montague, Warger, & Morgan, 2000). Furthermore, word problem solving also requires that students use prior knowledge of content and apply this knowledge to a real world situation. For students with LD, this is particularly challenging as it involves higher order thinking and knowledge transfer (Hudson & Miller, 2006). Only after students understand the problem and translate it into a mental, schematic, and/or algorithmic model can they actually compute the problem and finally check that the answer is both reasonable and accurate.

As mathematical expectations have increased, so has the role of problem solving in math curricula throughout the nation. Problem solving has been listed as one of the National Council of Teachers of Mathematics (NCTM) standards for all grade levels (NCTM, 2000) and as one of the standards for mathematical practice highlighted in the Common Core State Standards for Mathematics (CCSS-M, 2011). Since these standards direct what is taught in schools, it is no surprise that both statewide (e.g., Florida Comprehensive Assessment Test) and nationwide (e.g., National Assessment of Educational Progress) assessments contain substantially more problem-solving items. Moreover, widespread implementation of Common Core standards has led to a greater emphasis on conceptual understanding of mathematical topics. As a result, word problems in particular are being used more frequently in classrooms for both instructional and assessment purposes, as this type of problem solving facilitates and necessitates deep conceptual understanding of the core concepts. Thus, it is crucial that we provide for students with LD the supplemental instruction on underlying skills necessary for word problem-solving success.

Though problem solving can take many forms (e.g., nonroutine problems, real-life problems), for the purposes of this article, we focus on math word problems, the most common problem-solving task in the math curriculum. Research on the math word problem-solving skills of students with LD corroborates the observations Mrs. Wilson noted above; that is, this population often demonstrates deficits in problem representation strategies (Krawec, 2014; Montague & Applegate, 2000; van Garderen & Montague, 2003). Problem representation is the initial phase in the problem-solving process, whereby the solver translates the problem into a personally understandable form and integrates the relevant information

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in the problem into a mathematical, visual, or mental model (Mayer, 1985, 2013). Problem representation strategies, thus, include schema activation, paraphrasing, as well as creating visual representations. Since accurate representation of the problem is crucial to solution accuracy (Pape & Tchoshanov, 2001; Silver, 2000), and being able to model with mathematics is another Common Core standard for mathematical practice (CCSS-M, 2011), instruction in word problem solving should emphasize the skills embedded within problem representation. However, a strategic instructional approach that focuses on representation is rarely the case in mathematics classrooms because while teachers are typically aware of the significance of representation, they may, like Mrs. Wilson, be unequipped to effectively teach it. The purpose of this article is to discuss the importance of creating accurate mathematical models of word problems prior to solving. We present an instructional approach focusing specifically on one type of model (number lines) that, within a comprehensive problem-solving routine, may improve students' problem-solving proficiency.

VISUAL-SPATIAL REPRESENTATIONS AND PROBLEM SOLVING

Poor representational ability has been linked to poor word problem-solving performance for students with and without disabilities (e.g., Hegarty & Kozhevnikov, 1999; Krawec, 2014). Not only do students with LD generate fewer representations when solving novel word problems than their peers without disabilities, but when these students do generate representations, the representations usually do not depict the spatial relationships among the elements (i.e., how the elements interconnect), which is necessary to understand and then solve the problem (van Garderen & Montague, 2003). Rather, they typically draw pictures that represent the context of the problem without including the critical information needed to solve the problem (i.e., pictorial representations). However, when they do create schematic, accurate representations of the problem, there is a strong positive correlation with solution accuracy (Krawec, 2014). The use of representational strategies to solve word problems has demonstrated success: Zhang and Xin's (2012) meta-analysis of word problem-solving interventions ranked problem representation techniques as the most effective type of intervention for students with LD.

Visual representations, a specific type of problem representation strategy, provide students with a reference to which they can return (thereby reducing the demands on their working memory), allow for self-regulation during the problem-solving process, assist with teacher monitoring and error analysis, and also increase student motivation (van Garderen, 2006). Moreover, visuals can help students develop conceptual understanding of the core concepts contained in the problem (Jitendra, DiPipi, & Perron-Jones, 2002). Consequently, research consistently supports the idea that structured lessons on the use of visual representations can improve the problem-solving performance of students with LD (e.g., Fuchs et al., 2009, 2010; Griffin & Jitendra, 2009; Jitendra et al., 2002; van Garderen, 2007). Griffin and Jitendra (2009) found that schema-based instruction (SBI), which requires students to map the problem information onto a relevant schematic diagram, improved the problem-solving performance of students in third grade. Likewise, Fuchs et al. (2010) found that teaching students strategies based on problem types improved second-grade students' problem solving on addition and subtraction items. Yet while these results are promising, neither study considered generalization of the strategy to more complex problems of the same nature (i.e., multiple steps, all four operations). Similarly, Jitendra et al. (2002) developed and tested an intervention using SBI to teach eighth grade students a strategy to solve proportional reasoning problems. Although a critical component of the curriculum, these types of problems are very specific and the strategy could only be generalized to more complex problems of the same type. As students with LD benefit more from learning, a small number of strategies that are broadly applicable versus learning many strategies that are applied to specific tasks (Gersten et al., 2009; van Garderen, Scheuermann, & Jackson, 2012), finding a representation model that is applicable to a wide range of problems should be the goal for instructing students with LD.

In addition to the need for a visualizing strategy to be generalizable, instruction in visualizing should not only target what a representation is, when and how representations should be used, and how to generate a representation for a given task (van Garderen et al., 2012), it should be taught in conjunction with other evidence-based instructional components (Gersten et al., 2009; Zhang & Xin, 2012). Consequently, while the focus of this article is on how teachers can teach students to use the problem representation strategy of visualizing through the use of number lines, it must be noted that this strategy is meant to be embedded within a comprehensive problem-solving routine (Montague, 2003) that incorporates other cognitive strategies (e.g., estimating and checking) and instructional components (e.g., explicit instruction and metacognitive strategies). Because of the particular difficulty teachers have met in trying to teach effective visualizing strategies (e.g., Krawec & Montague, 2014), we focus only on the particular cognitive strategy of visualizing and refer readers to the work of Montague (2003) for an in-depth description of the comprehensive problem-solving routine, of which visualizing is a critical part.

NUMBER LINES AS A PROBLEM-SOLVING STRATEGY

Number lines, a mathematical model very familiar to most students, are one such type of representational model that is widely applicable across concepts as well as grades. While number lines have been used in the classroom for many years in the teaching of numerous topics (e.g., measurement, time, fractions, and estimation), its use as an aid to word problem solving has been extremely limited in both research and practice. Although some curricular textbooks [e.g., *McDougal Littell middle school math: Course 1* (Boswell, Kanold, Larson, & Stiff, 2004), *Mathematics applications and concepts: Course 2* (Bailey et al., 2004), and *Go math! Florida* (Adams et al., 2011) textbook] do sometimes model visual representations of word problems using number lines, these textbooks simply prompt students to "draw

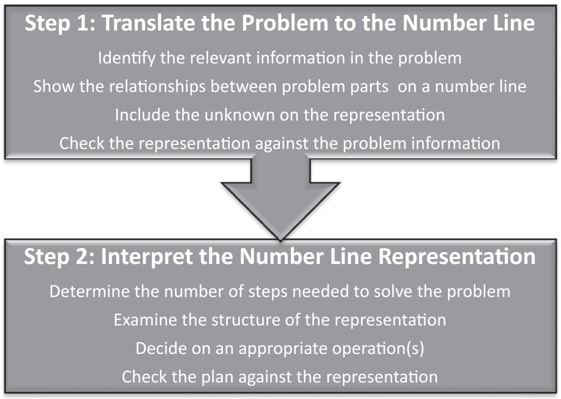


FIGURE 1 Advance organizer for instruction on using number lines to solve word problems.

a diagram showing the relationships" and/or to "use the diagram to solve the problem" (Bailey et al., 2004, p. 302); there is no explicit instruction in the use of number lines for this purpose. Most students, especially those with LD, do not benefit from such general prompts and need to be explicitly taught how to reconstruct the problem information in a number line format. Thus, although students may be able to interpret a representation provided in the textbook to solve a problem, they are rarely able to generalize the strategy to construct similar representations when solving subsequent word problems. Pedagogically, this suggests that students need systematic instruction in the generation and use of number line representations for story problems for the strategy to be truly beneficial.

Since number lines are a type of schematic representation (i.e., they illustrate the spatial relationships among the parts of the problem), they actively aid students in the problem-solving process (Hegarty & Kozhevnikov, 1999). In Mrs. Wilson's situation, her students drew elaborate pictures to represent each of the buildings in the problem, but did not include the distance relationships between the buildings. Consequently, these pictorial illustrations did not give the students any clues about how to solve the problem and were actually detrimental as a solving strategy. For this reason, students with LD should be taught how to independently construct meaningful schematic representations like number lines in order to become proficient problem solvers so that when faced with a difficult problem like the library problem, they have an effective strategy to apply.

Once the number line is accurately drawn to represent the word problem, it facilitates the organization of the student's mathematical thought processes and helps him/her determine a plan to solve the problem. The goal of using these representations is that students will "gradually shift their attention from thinking about the problem situation to thinking about the mathematical relations between the numbers involved" (Skoumpourdi, 2010, p. 2). Number lines make this process possible by greatly decreasing the amount of text that students must interpret before they can execute their solution plan (Kolloffel, Eysink, Jong, & Wilhelm, 2009; Skoumpourdi, 2010). After examining the visual relationships among the problem parts and the placement of the unknown in comparison to the given problem information, students are able to decide how many steps are needed to arrive at the final solution and what operation(s) should be used. As a result, number lines allow for observation of students' thinking processes, and thus teachers can quickly recognize and correct any errors in thinking that may occur (Bobis, 2007). Moreover, unlike other models, such as SBI, number lines are flexible and coherent, allowing students to use them throughout grades for increasingly complex problems of various types.

The following sections describe the steps of the visualizing strategy using number line representations for word problems. Instruction in this strategy should be divided into two phases: (1) translating the problem to the number line, and (2) interpreting the number line representation (Skoumpourdi, 2010). Once both skills are explicitly addressed and mastered in the classroom, the problem-solving strategy becomes a great tool for students to apply as needed. As stated previously, using number lines to visually represent word problems requires careful and systematic teaching before students become adept and improve their problem-solving skills. First, students must develop several subskills before they are able to confidently and accurately draw and interpret number lines to solve word problems. For this reason, content, sequence, and instructional approach must be considered with visual representation instruction for students with LD. Figure 1 shows an advance organizer that can be used to guide the representation lessons.

NUMBER LINES: INSTRUCTIONAL CONTENT

Step 1: Translating the Problem

Before solving a math word problem, students must understand the problem's key features linguistically (i.e., comprehending the problem through a verbal translation of its important information), which is especially challenging for students with language or reading difficulties, as well as mathematically (i.e., identifying the necessary algorithm to solve the problem). A number line representation of a word problem bridges the gap between linguistic understanding and mathematical understanding by depicting the problem information in an abstract but connected way. When translating a word problem onto a number line, three important components must be included in the representation: (1) the relevant information that was given in the problem, (2) the interrelationships or connections among that information, and (3) the goal or the question(s) in relation to the other information. Each of these components is equally important if students are to use the representation as a tool to determine a plan to solve the problem. Thus, it is necessary that, during instruction, teachers focus on these three components of the representation explicitly, so that students truly understand how to identify this information in the problem, incorporate it into the representation, and then translate that representation into a means of solving the problem.

3. Visualize (a picture or a diagram)

Ask: Does the problem consist of equal or unequal parts?

Say: Draw a diagram (number line) using all the important information.

Say: Include the goal on my diagram.

Ask: Does the picture fit the problem?

Ask: Can I see how the problem information connects to each other?

Check: The picture against the problem information.

FIGURE 2 Sample cue card for visualizing which includes self-instructing, self-questioning, and self-checking metacognitive prompts.

Identifying the Relevant Information

First, students should be taught how to extract the relevant information from the problem, by paraphrasing the problem (i.e., putting the problem in their own words). Students explicitly state what they are looking for and what information is already known from the problem, beginning with what the problem is asking. This process requires that students use their linguistic skills to comprehend and make sense of the problem by identifying the critical pieces of information. Students should underline the important information during this step to serve as a memory aid when retrieving this information for the representations later on. In addition, all irrelevant information should be excluded by crossing out, because placing extraneous information on the number line will result in either an inaccurate representation of the problem, leading to the wrong answer, or a distracting representation of the problem with accurate but unnecessary information, which leads to an increased cognitive load and a decreased likelihood of obtaining the correct answer (Kolloffel et al., 2009). For this reason, self-monitoring checks are built into all phases of number line instruction, so that students are taught how to identify and then correct their errors (Figure 2).

Showing Relationships among Problem Parts

Once students have identified the important information, they can begin to construct the number line representation for the problem by first determining what interrelationships exist between the elements of the problem (i.e., the relevant information that must be used to solve the problem). This component is arguably the most important, as these interrelationships will provide the students with a framework for solving the problem; it is also the most difficult component for students with LD to grasp (van Garderen, 2007). Thus, instruction needs to be very explicit and detailed, with numerous opportunities for practice. Students must first decide if the problem is made up of distinct/dissimilar elements (i.e., addition and subtraction problems) or if the problem consists of a certain number of equal/repeating groups (i.e., multiplication and division problems). At this stage of instruction, it is beneficial to provide students with several examples of each problem situation and a think-aloud model (i.e., a verbalization of the thought processes) of how to recognize each situation. For example:

> *Problem: The radio station plays 5 songs every hour. How many hours does it take to play 450 songs?*
> "I am trying to figure out how long it takes to play 450 songs. I know that every hour, 5 songs are played. Since the **same** number of songs are played every hour, that means that I have equal 'groups' with 5 songs in each group, or hour."

Once students are able to recognize the problem situation, they can determine representation features (e.g., "I am going to draw a line that represents songs played as time passes. So I am going to label the top of the line as songs and the bottom as time. Then I will split the line into equal-sized sections to represent each hour that passes. Since there will probably be a lot of hours, I am just going to draw a few and then put '. . . ' to show that this pattern keeps repeating. The entire line will represent 450 songs, since that is the total amount of songs played in this problem. I'll put that at the top, where I have my total, or overall amount."). From here, students incorporate the relevant information onto the number line, following the structural features they identified. After placing all the corresponding numerical information in the appropriate places on the number line, students should check to ensure that they have used all the relevant information underlined during the previous step.

Including the Problem Question(s)

Finally, students must place the unknown of the problem on the representation. This step involves its strategic placement in the form of a question mark, signifying what the students are looking for. Although this component was already stated during the paraphrasing phase, students must now connect it to the information they already know in order to place it correctly on the number line. Students ask themselves, "Am I looking for the total?" If the answer is yes, then the unknown is placed at the top of the representation, indicating that the problem is asking for the combined value of all the information on the number line. If the answer is no, then the unknown lies somewhere on the number line itself and its position should be determined by examining what still is missing on the line (e.g., "I am not looking for my total; I already know that 450 songs are played in all. I want to find how many hours it takes. I am going to put a question mark at the bottom of the last section on my number line to show that I am looking for the number of hours that passed."). It is important to note that for multistep problems, there is more than one unknown and thus, more than one question mark should be placed in the representation. We teach students to circle their final question mark to help sequence their problem-solving plan.

Once the number line has been constructed, students are taught to perform several checks to ensure that the representation is accurate and informative. Following a cue card with prompts, students first check that they included all the underlined information on the number line. Then they check that the picture fits the problem (i.e., the location of the question mark matches the question being asked) and finally, they check that they can see how the information relates.

Step 2: Interpreting the Number Line

After constructing an accurate number line model of the problem using the steps outlined in the previous section, students must now examine and interpret this model in order to develop a plan to solve the problem. Since the number lines are completely student-generated, the representations are very meaningful and reduce the difficulty of the problem (van Garderen, 2007). As a result, students should be able to grasp this step more readily and instruction may move at a faster pace.

First, students must decide how many steps are needed to get to the solution by noting how many unknowns are in the representation. For multistep problems, students must determine which unknown represents the final answer, so that the computation may be ordered accordingly. Then, for each solution step, students examine the structure of the number line and the position of the unknown, and link this information to the conceptual meanings of the four operations. In other words, students should use the relationships shown on the representation to decide on an appropriate operation. For example, if the number line is divided into a specific number of equal groups and the unknown is the combined amount (or total) of all the groups, then students should recognize that multiplication is the necessary operation. See Figure 3 that displays a student-generated number line for each operation.

NUMBER LINES: SEQUENCING INSTRUCTION

Student progress should be monitored consistently throughout implementation of this strategy so that instruction may be adapted and tailored according to individual student needs. In order for students to really acquire and maintain the use of this visualization skill, instruction should be provided in stages of gradually increasing complexity. That is, students should be introduced to the strategy using simple one-step arithmetic word problems. Only after they have demonstrated mastery on the use of the strategy for these types of problems should the instructor introduce the strategy for two-step problems and eventually more difficult multistep problems or problems containing more advanced math concepts such as fractions or percentages. Figure 4 shows how Mrs. Wilson used the strategy in her classroom to solve the library problem, which is a multistep problem containing fractions.

To determine the pace of instruction, students' visual representation as well as their actual problem-solving ability should be measured periodically. While teachers should consistently assess students' understanding informally using classroom discussion where students justify their number lines, compare models, and describe solution paths, they can also use a rubric to score students' use of number lines, based on the previously discussed necessary components of an effective representation. (For a rubric example, see Figure 5.) Using this rubric will also help teachers identify the specific component(s) with which students struggle. Only when the number lines are accurately drawn and students are able to explain their thinking processes should teachers increase the complexity of the problems.

In addition to the sequencing of problem difficulty, it is important that teachers sequence the skills required for effective visualizing. Thus, visual representation instruction should follow a review of the critical preskills that will enable students to effectively carry out the instructional content they are taught. Students must demonstrate conceptual understanding of two constructs: the number line and the four arithmetic operations.

Conceptual Understanding of the Number Line

To use number lines effectively, students must first understand two underlying ideas regarding this mathematical tool: (1) it is a graphic and thus, each position marked represents important information, and (2) it is a measurement model, so the numbers marked represent length rather than arbitrary points (Diezmann, Lowrie, & Sugars, 2010). This means that students should be taught to think about the distance/amount between the labeled segments and the accumulation of the quantities as the line progresses when interpreting number lines. Teachers should therefore provide explicit instruction beforehand, focusing on the measurement aspect of the number line while pointing out that it is not simply a counting model (Diezmann et al., 2010). Though students are not required to place this explicitly in their model, they learn that the far left tick of the number line represents zero and the farthest right tick (also indicated by the connecting line that encompasses the entire number line above) represents the whole amount or total (i.e., sum or product). Number line estimation tasks can ensure that students truly grasp this concept. For example, students may be given a number line with certain end points and asked to estimate where a certain number should be placed on the line, or students may be given a structured number line with labeled end points, A and C, and a point on the line, B, and a problem that reads "On the road shown above, the distance from A to B is 60 kilometers. What is the distance from A to C?" Again, challenging the students to explain their thinking processes while using number lines with word problems will not only be very insightful to teachers, but can also facilitate students' self correction of errors and ultimately ensure deep conceptual understanding.

Conceptual Understanding of Operations

It has been found that many students with LD lack conceptual understanding of the four mathematical operations, yet this is a critical prerequisite for effective use of the strategy described in this article. In order to interpret the number line drawn, students must understand the meaning of the operations (e.g., that to find a total consisting of distinct parts, they need addition, or that to find how many equal groups of a certain size can be made from a specific overall amount, they need division). Without conceptual understanding of the operations, students are unable to come up with an accurate

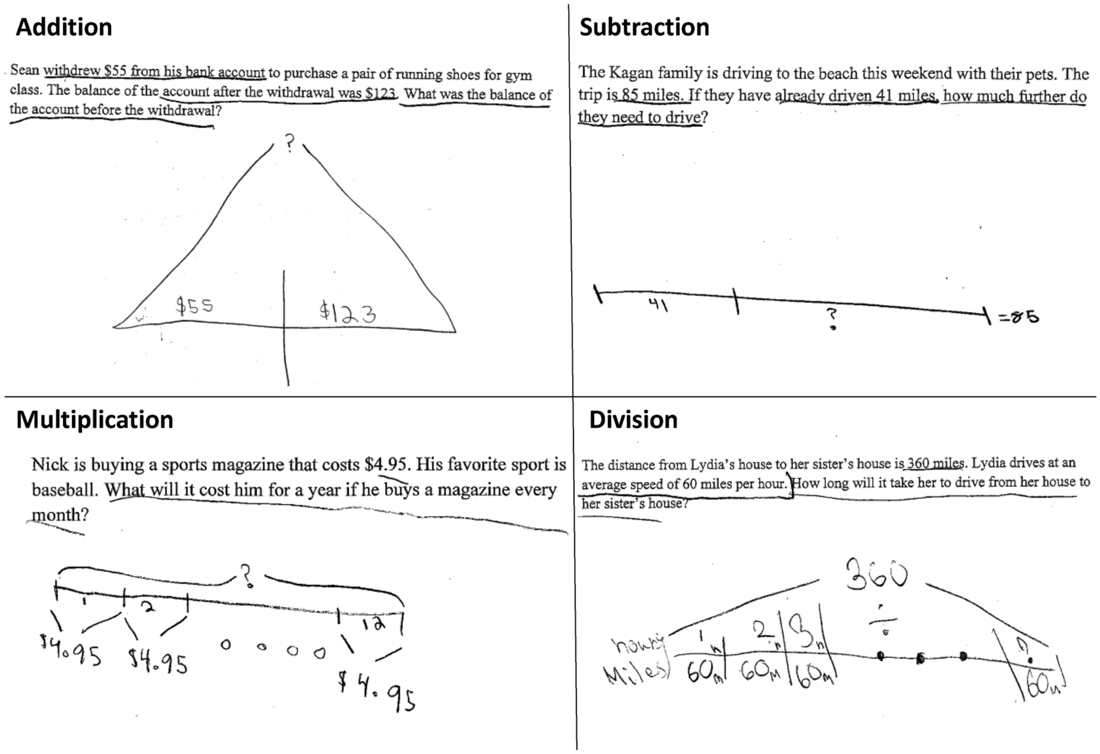


FIGURE 3 Examples of student-generated number lines reflecting each of the four operations.

plan to solve the problem by observing the relationships in the representation, and rely instead on less effective strategies such as using key words from the problem. In math problem solving, the value of the representation to the student is a function of his/her knowledge of the four operations. For this reason, we advise teachers to incorporate a thorough review of the concepts of operations before actual problem-solving instruction. Figure 6 depicts a decision tree, along with the operation definitions, which guide this lesson and which students may use when interpreting their number lines.

NUMBER LINES: INSTRUCTIONAL APPROACH

As mentioned previously, the use of number lines for word problem solving should be taught using cognitive strategy instruction, an instructional model that incorporates modeling, verbal rehearsal, goal setting, and gradual release of responsibility of learning (Krawec et al., 2013). Instructors should therefore emphasize reflectivity so that students learn how to self-correct and navigate the problem-solving process nonlinearly. This involves incorporating metacognitive prompts into the lesson, which cue students to focus on specific points within the problem, ask themselves specific questions to guide their thinking, and check their work throughout the process. Cue cards/checklists are helpful to assist students with the process but should be phased out over time, as the goal is for the students to internalize the skills and begin to use the processes automatically and flexibly. (Refer back to Figure 2 for an example of a cue card with metacognitive prompts.)

Supplemental Instruction

Many students, particularly those with LD, may struggle at the outset with the abstractness of the number line representation and so may benefit from the use of a concrete-representational-abstract (CRA) sequence of instruction (Mancel, Miller, & Kennedy, 2012; Mercer & Miller, 1992). Initially, these students should be presented with a more concrete model that uses the same basic underlying format as the number line, thus attaching meaning to solution strategies in a similar way. An example of a concrete lesson in number lines is provided in Bay's (2001) article on number line usage. Bay used a life sized number line in her classroom to develop number sense in her students, setting certain end points for the line, and having students place number cards in an appropriate location on the line. By continuously changing the end points of the line, students learned about the relative positions of numbers, including fractions and very large numbers, and that the line represents a certain length. This activity allowed students to access the measurement aspect of the number line as they

"The mall is 12 miles from your home. Your school is one third of the way from your home to the mall. The library is one fourth of the way from the school to the mall. How far is the library from home?"

Now I will **visualize** the problem using a number line representation. First, I ask myself, *Does the problem consist of equal or distinct parts?* Hmm, this problem is about distances between certain places. None of these distances seem to be equal, so I believe that this problem consists of distinct elements.

Now I say to myself, *Draw a number line using all the important information*. OK, so first I will draw a line that will represent the distance between my home and the mall, since everything else seems to be located in between. I will label one end of my line 'HOME' and the other end 'MALL.' I know that this entire distance is 12 miles, so let me put that at the top of this line to show that this entire line is really 12 miles. Next, let me put the school in place. Since school is one third of this distance from home to the mall, let me separate my line into 3 equal sections (or thirds). Then school should be located at the end of the first section since this is one third of the line. Now let me see where I should put the library. I know that the library is one fourth of the way from school to the mall. I don't want to make my line too confusing, so let me draw a portion of this line directly below this one, from school to mall only. Now, I will separate this line into 4 equal sections (or fourths) and put the library at the end of the first section since it is located one fourth of the way. Ok, I think I have all the important information.

Next, I say to myself, *Include the goal or question*. OK, the question asks for the distance between home and the library. Let me bring a dotted line from the second line up to my first so that I can see where the library is on my original number line. Looking at my representation, the library is located just past the school. So let me put a question mark over the distance from home to the library since this is ultimately what I need to find. But it looks like I need to find some other information first. SO I will put a smaller question mark over the distance from home to school and then another one over the distance from school to the library. Once I find out what these two distances are, I can find out the distance from home to the library.

Now I ask myself, *Does the representation fit the problem*? Yes, because it shows that I am looking for the distance between my home and the library which is what the problem is asking. *Can I see how the problem information connects to each other?* I can see that the school is one third of the way between the home and the mall, and the library is one fourth of the way between the school and the mall. And the total distance from home to the mall is 12 miles. Yes, I can see all the connections.

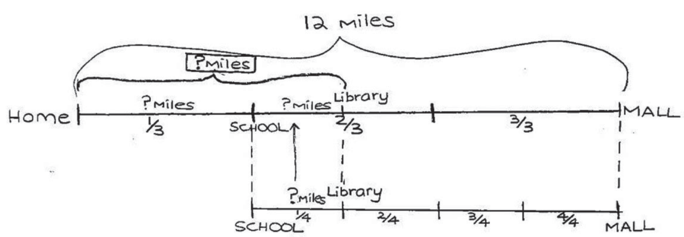


FIGURE 4 Mrs. Wilson's use of the number line strategy for the library problem.

looked at and compared distances between certain numbers. This "live" number line is a viable way of reinforcing these foundational concepts if students initially struggle with the two-dimensional representation. The representational use of number lines is the focus of this paper, but it is possible to increase the explicitness of the number line in order to help introduce students to the strategy. This model should still include the three components necessary for an accurate representation of the problem (i.e., relevant information, relationships among problem parts, and the question). There should also be extensive labeling of the problem parts, so that students can observe what each number represents along with some pictorial elements incorporated onto the model to more clearly convey the schematic components within the problem context; this includes marking the far left tick as zero and the far right tick as the total or overall amount. In order to make the number line as explicit as possible, each individual unit should be marked on the number line so that "jumps" can be drawn to highlight the accumulation of units along the line. However, the explicitness of this number line is not practical (or feasible) for large quantity problems; thus, it should be used for initial instruction and concept development in the use of the number line, and then phased out and replaced by more abstract models. Figure 7 shows an example of a highly explicit number line for a particular problem.

Case Study: Maria

Maria*, a sixth-grade student with LD, was instructed on the use of this strategy for the purpose of a single-subject research project (J. Krawec, in preparation) primarily designed to evaluate the effectiveness of a modified math problem -solving intervention (Montague, 2003). Although Maria received the full intervention (i.e., instruction on reading, paraphrasing, visualizing, hypothesizing, estimating, computing, and checking), the emphasis of the intervention was on paraphrasing and visualizing. This section specifically focuses on

Components	0	1	2	Comments
Relevant information included	No information	Some information	All information	
Relevant information accurately related	Nothing accurately related	Some accurately related	All accurately related	
Irrelevant information excluded	Any irrelevant information	-------	No irrelevant information	
Question included	Not included	Included but inaccurate	Included and accurate	
			TOTAL POINTS	

FIGURE 5 Example of a rubric that may be used to score students' visual representations, based on the four necessary components for a meaningful representation.

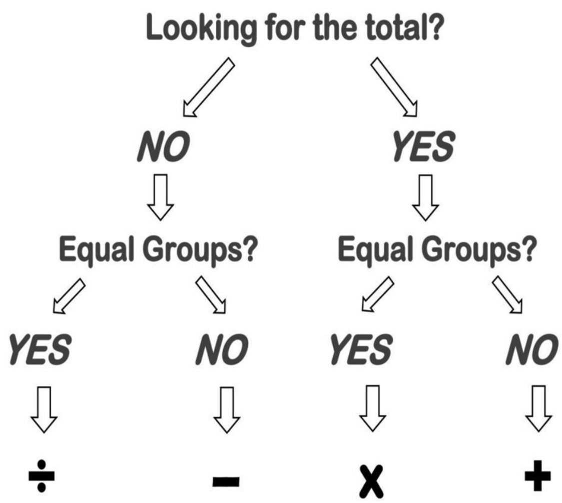


FIGURE 6 Decision tree that students can use to determine which operation is needed to solve a problem.

Maria's development in the visualizing phase, during which she learned to use number lines as a means to represent word problems. She had previously mastered the paraphrasing phase, after which she accurately identified relevant information and the question, excluded irrelevant information, and put the problem in her own words. A trained research assistant conducted sessions during a 35-minute homeroom period approximately three times a week. After each session, the student was given a process measure consisting of four arithmetic word problems to assess her progress on the process being taught. Once the student demonstrated mastery of the process (at least 75% on 3 of 4 consecutive measures), a problem-solving measure was administered before moving on to instruction of the next process.

Figure 8 shows the progression of Maria's visual representations at baseline, during visualization instruction, at mastery after five lessons specifically targeting visualizing, and finally at maintenance at 2- and 4-month time points. Prior to receiving instruction, Maria drew inefficient, developmentally immature, and incorrect representations (note that she drew 47 circles, not 50). After one instructional lesson, she was able to draw the correct basic structure of the number line, but unable to depict the correct relationships among components (note that there should be 17 equal sections on the line with a question indicating how many miles in one section). However, when she met mastery on the visualizing strategy (after five instructional lessons), she included all relevant information accurately and explicitly related, with the unknown correctly placed amidst the other parts of the problem. While her drawing from lesson two is structured schematically rather than pictorially, it is clear that Maria did not yet understand how to accurately depict the interrelationships of the problem parts. However, with subsequent lessons emphasizing this skill, she was able to consistently create accurate schematic representations on both one- and two-step problems. Moreover, Maria's problem-solving accuracy scores demonstrated an upward trend over time, with a score of 100% following visualizing mastery. Two months following intervention, visualizing proficiency using number lines was maintained at mastery level. Four months following intervention, however, Maria was not able to maintain the use of accurate number line models; though she was able to identify that she needed to find the total or overall amount, she could not accurately place the problem information on the number line and, subsequently, could not answer the problem. That her visualizing skills were not maintained four months later suggests that, unsurprisingly, periodic review of the strategy and opportunities for use may be necessary to maintain student proficiency over time.

Like Maria, the other nine students in the study improved the accuracy and the schematic nature of their word-problem representations after just one instructional session in this skill, with most taking an average of five lessons to master the skill. Furthermore, the majority of students saw improvements in their problem-solving scores after these visualizing lessons, further validating the use of this strategy. Again, it must be noted that these representations were taught in the context of a larger problem-solving strategy; the use of number lines alone may not improve students' performance

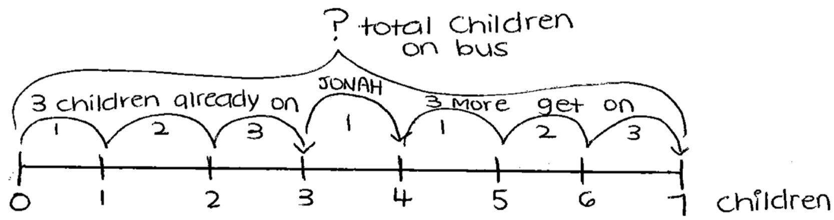
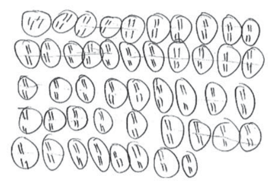
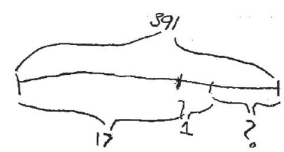


FIGURE 7 Example of a more explicit number line for the following problem: “Jonah got on the bus at 8 o’clock in the morning. When Jonah got on the school bus there were 3 other children on the bus. At the next stop 3 more children got on. How many children were now on the bus?”

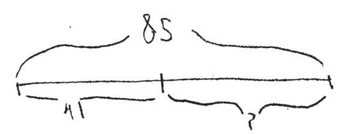
a)
A roller coaster at the state fair has cars that hold 4 people each. Adam and his family filled the first car. How many people can ride at the same time if there are 50 cars?



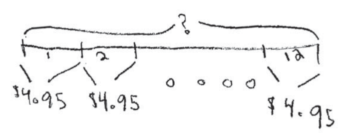
b)
Dr. Rodriguez drove to Disney World with his family this past weekend. Dr. Rodriguez drove 391 miles on 17 gallons of gasoline. At this rate, how many miles could he drive on 1 gallon?



c)
The Kagan family is driving to the beach this weekend with their pets. The trip is 85 miles. If they have already driven 41 miles, how much farther do they need to drive?



d)
Nick is buying a sports magazine that costs $4.95 because it covers baseball, his favorite sport. What will it cost him for a year if he buys a magazine every month?



e)
Devon went to the movies on Friday. He bought 3 movie tickets and 1 large popcorn. Each movie ticket was $5 and the large popcorn was $4. How much did Devon spend altogether?

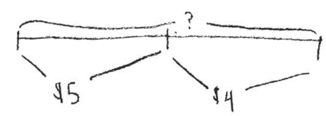


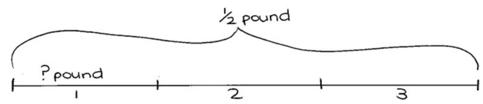
FIGURE 8 Progression of Maria’s visuals: (a) baseline, (b) lesson 1, (c) lesson 5, (d) 2-month maintenance, and (e) 4-month maintenance.

on math problem solving, but it will provide them with a flexible, concise tool that will help them plan, carry out, and solve math word problems.

LIMITATIONS

While the use of number lines shows promise as a flexible, manageable tool for students’ use during word problem solving, there are two important limitations that should be noted. First, the number lines that students generate to solve word problems may not be a conceptually perfect representation of a number line (e.g., see Figure 3, addition). As described above, students must receive conceptually accurate, explicit instruction in the structure and meaning of number lines before using them as models for word problems. Yet, as students increase proficiency, they will modify the model as it reflects their understanding. Because the number line is a permanent product, teachers have the opportunity to address and correct misconceptions, but should also be aware that some modifications may just represent (as in other visual representations of problems) a student’s “making it his own” without detracting from the accuracy of the model. We recognize that this is a slippery slope, but agree with Cleveland and McGill (1985) that “the power of a graph is its ability to enable one to take

How much chocolate will each person get if 3 people share ½ lb of chocolate equally?



If it took 7 hours to mow 4 lawns, then at that rate, how many lawns could be mowed in 35 hours?

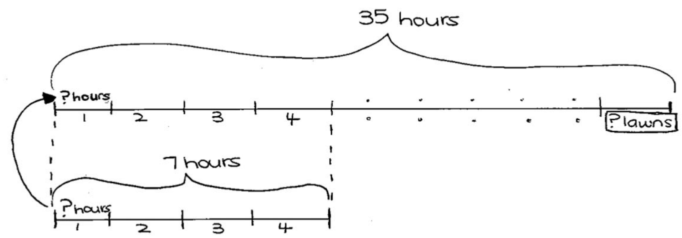


FIGURE 9 CCSS-M word problem example from fifth grade and from sixth grade with accompanying number line models.

in the quantitative information, organize it, and see patterns and structure not readily revealed by other means" (p. 535). If students can demonstrate that the critical components of the visualization are in place (i.e., identifying relevant information, showing relationships among problem parts, and including the question), their visualizations has merit.

The second limitation lies in the fact that the number line representational strategy cannot be applied to the full range of problems students see in the curriculum (e.g., division with remainders). We utilized this strategy with fifth and sixth grade students with LD. In the fifth and sixth grade curriculum, students are expected to solve word problems related to the multiplication and division of fractions as well as those involving area and volume (CCSS-M, 2011). Before students can solve these complex word problems, though, they must be able to solve similar problems with whole numbers. As such, the bank of word problems in our study was limited to one- and two-step problems using the four operations with whole numbers. However, the number line model can still represent more complex problems. See Figure 9, which shows a fifth and a sixth grade word problem pulled directly from the CCSS-M as well as a number line model that could be used to solve each of the problems.

CONCLUSION

The ability to make a mental or visual representation of a math word problem is a critical strategy to support problem-solving proficiency (Organization for Economic Co-operation and Development [OECD], 2013). Visual representations that are too structured leave little room for flexibility as problems increase in complexity. Yet asking students to create schematic representations that accurately depict problem information is difficult, particularly for students with LD who benefit from more structured, explicit prompts. Even teachers struggle to teach students how to represent problems in a way that will help students develop a viable solution path (Krawec & Montague, 2014). Number lines provide an accessible and flexible model for students to use when solving word problems. When teachers attend to the instructional content, sequence, and approach of math problem-solving instruction as described in this article, students (particularly those with LD) will find number lines to be a very supportive tool to develop proficiency in math word problem solving. Although number lines will not always be an appropriate representational model across the entire range of problem types that students may encounter in their academic learning (e.g., though they can be used for area problems, they are not the logical structure; they are not ideal for division with remainders), it is clear that understanding how to construct and interpret number lines will facilitate the development of more sophisticated schematic diagrams to solve these more advanced problems, while simultaneously reinforcing students' conceptual understanding of operations and, more broadly, number sense.

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